

RAY OPTICS IN THE FIELD OF NON-MINIMAL DIRAC MONOPOLE

A.B. Balakin¹ and A.E. Zayats²

*Department of General Relativity and Gravitation,
 Kazan University, Kremlevskaya str. 18, Kazan 420008, Russia*

Based on the analogy with non-minimal $SU(2)$ symmetric Wu-Yang monopole with regular metric, the solution describing a non-minimal $U(1)$ symmetric Dirac monopole is obtained. In order to take into account the curvature coupling of gravitational and electromagnetic fields, we reconstruct the effective metrics of two types: the so-called associated and optical metrics. The optical metrics display explicitly that the effect of birefringence induced by curvature takes place in the vicinity of the non-minimal Dirac monopole; these optical metrics are studied analytically and numerically.

1. Introduction

The Dirac monopole as a specific static spherically symmetric solution to the minimal Einstein-Maxwell equations has become a subject of discussion in tens papers, reviews and books (see, e.g., [1, 2, 3, 4, 5, 6, 7]). The motion of massive and massless particles, which possess electric charge or are uncharged, is studied in detail (see, e.g., [8, 9] and references therein). In the paper [10] we introduced and discussed the $SU(2)$ symmetric Wu-Yang monopole of the new type, namely, the non-minimal monopole with regular metric. Since the non-minimal Wu-Yang monopole is effectively Abelian, it is naturally to consider the corresponding analog of that solution in the framework of non-minimal electrodynamics. Mention that non-minimal models with magnetic charge have been discussed earlier (see, e.g., [11, 12]), but the exact analytical regular solution obtained here as direct reduction to the $U(1)$ symmetry is the new one.

The second novelty of the presented paper is the investigation of photon dynamics in the vicinity of the Dirac monopole accounting the curvature coupling of the gravitational and electromagnetic fields. In the presence of non-minimal interaction (induced by curvature) the master equations for electromagnetic and gravitational fields in vacuum can be rewritten as the master equations in some effective anisotropic (quasi)medium [13, 14]. This means that two effective (optical) metrics can be introduced [15, 16, 17], so that the photon propagation in vacuum interacting with curvature is equivalent to the photon motion in the effective space-time with the first or second optical metric, depending on the photon polarization. Even if the real space-time has a regular metric, the optical metrics can be singular, admitting the interpretation in terms of the so-called “trapped surfaces” and “inaccessible zones” [18, 19]. We discuss this problem in Sect. 3. Numerical modeling of the photon orbits, presented in Sect. 4, supplement our conclusions.

2. Master equations and background fields

We consider the Lagrangian³

$$\mathcal{L} = \frac{R}{8\pi} + \frac{1}{2}F_{ik}F^{ik} + \frac{1}{2}\mathcal{R}^{ikmn}F_{ik}F_{mn} \quad (1)$$

to describe the background gravitational and magnetic fields in the framework of non-minimal Einstein-Maxwell model with the non-minimal susceptibility tensor

$$\mathcal{R}^{ikmn} = \frac{q}{2} \left[R(g^{in}g^{km} - g^{im}g^{kn}) - 12R^{ikmn} + 4(R^{im}g^{kn} + R^{kn}g^{im} - R^{in}g^{km} - R^{km}g^{in}) \right] \quad (2)$$

linear in curvature. As usual, R^{ikmn} is the Riemann tensor, R^{mn} is the Ricci tensor, R is the Ricci scalar, F_{mn} is the Maxwell tensor. This non-minimal susceptibility tensor can be obtained from the general one (see [14]), when the

¹e-mail: Alexander.Balakin@ksu.ru

²e-mail: Alexei.Zayats@ksu.ru

³Hereafter we use the units $c = G = \hbar = 1$.

coupling constants q_1, q_2, q_3 are chosen as follows $q_1 = -q < 0$, $q_2 = 4q$, $q_3 = -6q$. The ansatz for the space-time metric is

$$ds_{(0)}^2 = N(r)dt^2 - \frac{dr^2}{N(r)} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (3)$$

Mention that the choice $g_{tt}g_{rr} = -1$ in our ansatz is supported by the result, obtained in [10] for the given relations between the coupling constants q_1, q_2, q_3 . The equations of electrodynamics [15, 20]

$$\nabla_k H^{ik} = 0, \quad \nabla_k F^{*ik} = 0 \quad (4)$$

with constitutive equations

$$H^{ik} = C^{ikmn} F_{mn}, \quad C^{ikmn} \equiv \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) + \mathcal{R}^{ikmn} \quad (5)$$

are associated with the Lagrangian (1), where F^{*ik} is the dual Maxwell tensor and H^{ik} is the induction tensor. The equations (4) with (5) are satisfied identically, when the potential of static spherically symmetric electromagnetic field out of the point-like magnetic charge μ is of the form

$$A_k = \delta_k^\varphi A_\varphi = -\delta_k^\varphi \mu (1 - \cos \theta). \quad (6)$$

The corresponding strength field tensor has only one non-vanishing component

$$F_{\theta\varphi} = -\mu \sin \theta, \quad (7)$$

which does not depend on the non-minimal coupling parameter q . Thus, the well-known solution with the magnetic field of the monopole type

$$B^i \equiv F^{*ik} U_k = \delta_r^i \frac{\mu \sqrt{N}}{r^2}, \quad B(r) \equiv \sqrt{-B^i B_i} = \frac{\mu}{r^2}, \quad (8)$$

satisfies the *non-minimal* Maxwell equations (4), (5), (2). The equations for the gravity field

$$R_{ik} - \frac{1}{2} R g_{ik} = 8\pi T_{ik}^{(\text{eff})}, \quad (9)$$

obtained by direct variation procedure from the Lagrangian (1) are non-minimally extended, since the effective stress-energy tensor $T_{ik}^{(\text{eff})}$ has the form

$$T_{ik}^{(\text{eff})} = T_{ik}^{(0)} - q \left[T_{ik}^{(1)} - 4T_{ik}^{(2)} + 6T_{ik}^{(3)} \right]. \quad (10)$$

The quantities $T_{ik}^{(0)}$, $T_{ik}^{(1)}$, $T_{ik}^{(2)}$ and $T_{ik}^{(3)}$ are given by

$$T_{ik}^{(0)} = \frac{1}{4} g_{ik} F_{mn} F^{mn} - F_{im} F_k^m, \quad (11)$$

$$T_{ik}^{(1)} = R T_{ik}^{(0)} - \frac{1}{2} R_{ik} F_{mn} F^{mn} - \frac{1}{2} g_{ik} \nabla^l \nabla_l (F_{mn} F^{mn}) + \frac{1}{2} \nabla_i \nabla_k (F_{mn} F^{mn}), \quad (12)$$

$$T_{ik}^{(2)} = -\frac{1}{2} g_{ik} [\nabla_m \nabla_l (F^{mn} F_{n}^l) - R_{lm} F^{mn} F_{n}^l] - F^{ln} (R_{il} F_{kn} + R_{kl} F_{in}) - R^{mn} F_{im} F_{kn} - \frac{1}{2} \nabla^l \nabla_l (F_{in} F_k^{n}) + \frac{1}{2} \nabla_l [\nabla_i (F_{kn} F^{ln}) + \nabla_k (F_{in} F^{ln})], \quad (13)$$

$$T_{ik}^{(3)} = \frac{1}{4} g_{ik} R^{mnls} F_{mn} F_{ls} - \frac{3}{4} F^{ls} (F_i^{n} R_{knls} + F_k^{n} R_{inls}) - \frac{1}{2} \nabla_m \nabla_n (F_i^{n} F_k^{m} + F_k^{n} F_i^{m}). \quad (14)$$

We are happy to stress that with the ansatz (7) for the metric (3) the system of equations (9) transforms into one equation

$$rN' \left(1 + \frac{\kappa q}{r^4} \right) + N \left(1 - 3q \frac{\kappa}{r^4} \right) = 1 - \frac{\kappa}{2r^2} - 3q \frac{\kappa}{r^4}, \quad (15)$$

whose solution is

$$N(r) = 1 + \frac{r^2(\kappa - 4Mr)}{2(r^4 + \kappa q)}. \quad (16)$$

Here κ is the convenient constant, $\kappa = 8\pi\mu^2$. This solution is regular in the center ($N(0) = 1$) and satisfies the asymptotic condition $N(\infty) = 1$. If the mass M is less than its critical value $M_{(\text{crit})}$, where

$$M_{(\text{crit})} = \frac{r_*}{6} \left(4 + \frac{\kappa}{r_*^2} \right), \quad r_* = \frac{\sqrt{\kappa}}{2} \sqrt{\left(\sqrt{1 + \frac{48q}{\kappa}} + 1 \right)}, \quad (17)$$

the metric (3) with (16) has no horizons as in [10].

3. Electrodynamic description of the photon propagation

Linear electrodynamics allows us to consider the dynamics of test photons in terms of microscopic field strength f_{ik} and induction h_{ik} , which satisfy the equations

$$\nabla_k h^{ik} = 0, \quad h^{ik} = C^{ikmn} f_{mn}, \quad \nabla_k f^{*ik} = 0. \quad (18)$$

Clearly, the microscopic tensor f_{mn} is the analog of the macroscopic Maxwell tensor F_{mn} , describing the background electromagnetic field, and h^{ik} is the analog of the tensor H^{ik} (compare (4), (5) and (18)). We have to stress that the tensor of material coefficients C^{ikmn} in the formula (18) describing the photon propagation, is the same one as in (5) for the background field. Thus, the non-minimal interaction influences the photon by two ways: first, via the constitutive equations (see (5), (2)), second, via the gravitational field, whose potentials (16) depends on the parameter of non-minimal coupling q .

3.1. Associated metrics

The tensor C^{ikmn} can be represented algebraically as a quadratic combination of the so-called *associated* metrics $g^{im(\alpha)}$ [15, 17]

$$C^{ikmn} = \frac{1}{4\hat{\mu}} \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} \left[g^{im(\alpha)} g^{kn(\beta)} - g^{in(\alpha)} g^{km(\beta)} \right], \quad (19)$$

where $G_{(\alpha)(\beta)}$ and $\hat{\mu}$ compose a set of parameters, introduced in [17]. When the medium is spatially isotropic, this decomposition requires one associated metric; in the case of uniaxial spatial symmetry one needs two associated metrics [16, 17]. In the framework of our model vacuum interacting with curvature can be regarded as an uniaxial quasi-medium, where the radial direction is the selected one. Indeed, let us use the standard definitions of the dielectric permittivity tensor ε^{ik} , magnetic impermeability tensor $(\mu^{-1})_{pq}$, and tensor of magnetoelectric coefficients ν_p^m , given by [15, 20]

$$\varepsilon^{im} = 2U_k U_n C^{ikmn}, \quad (\mu^{-1})_{pq} = -\frac{1}{2} \eta_{pik} C^{ikmn} \eta_{mnq}, \quad \nu_p^m = \eta_{pik} C^{ikmn} U_n. \quad (20)$$

Here $U^k = \delta_t^k / \sqrt{N}$ is the velocity four-vector, associated with the magnetic charge (at rest), $\eta_{pik} = \epsilon_{pikm} U^m$ and ϵ_{pikm} is the Levi-Civita tensor. In the spherically symmetric model under consideration the tensor ν_p^m is equal to zero identically, other tensors have the following non-vanishing components:

$$\varepsilon_{||} \equiv \varepsilon_r^r = 2C^{rt}_{rt} = 1 + 2\mathcal{R}^{rt}_{rt}, \quad (21)$$

$$\varepsilon_{\perp} \equiv \varepsilon_{\theta}^{\theta} = \varepsilon_{\varphi}^{\varphi} = 1 + 2\mathcal{R}^{\theta t}_{\theta t} = 1 + 2\mathcal{R}^{\varphi t}_{\varphi t}, \quad (22)$$

$$\frac{1}{\mu_{||}} \equiv (\mu^{-1})^r_r = 1 + 2\mathcal{R}^{\theta\varphi}_{\theta\varphi}, \quad (23)$$

$$\frac{1}{\mu_{\perp}} \equiv (\mu^{-1})^{\theta}_{\theta} = (\mu^{-1})^{\varphi}_{\varphi} = 1 + 2\mathcal{R}^{\theta r}_{\theta r} = 1 + 2\mathcal{R}^{\varphi r}_{\varphi r}. \quad (24)$$

For the metric (3) with (16) we obtain

$$\mathcal{R}^{\theta\theta}_{r\theta} = \mathcal{R}^{r\varphi}_{r\varphi} = \mathcal{R}^{\theta t}_{\theta t} = \mathcal{R}^{\varphi t}_{\varphi t} = -\frac{r}{(r^4 + a^4)^3} [6qMr^8 - 3a^4r^7 - 24qMa^4r^4 + 5a^8r^3 + 2qMa^7], \quad (25)$$

$$\mathcal{R}^{rt}_{rt} = \frac{r}{(r^4 + a^4)^3} [12qMr^8 - 7a^4r^7 - 76qMa^4r^4 + 17a^8r^3 + 8qMa^8], \quad (26)$$

$$\mathcal{R}^{\theta\varphi}_{\theta\varphi} = \frac{r^4}{(r^4 + a^4)^3} [12qMr^5 - 5a^4r^4 - 20qMa^4r + 3a^8]. \quad (27)$$

A new parameter a with dimensionality of length is defined as follows: $a^4 = \kappa q$. Since $\varepsilon^\theta_\theta = \varepsilon^\varphi_\varphi$ and $(\mu^{-1})^\theta_\theta = (\mu^{-1})^\varphi_\varphi$, the medium can be considered as uniaxial. Moreover, due to the equality $\mathcal{R}^{\theta\theta}_{r\theta} = \mathcal{R}^{\theta t}_{\theta t}$ the relation $\varepsilon_\perp \mu_\perp = 1$ takes place. The tensor C^{ikmn} can be reconstructed algebraically as

$$C^{ikmn} = \frac{1}{2\hat{\mu}} \left\{ \left[g^{im(A)} g^{kn(A)} - g^{in(A)} g^{km(A)} \right] - \gamma \left[\left(g^{im(A)} - g^{im(B)} \right) \left(g^{kn(A)} - g^{kn(B)} \right) - \left(g^{in(A)} - g^{in(B)} \right) \left(g^{km(A)} - g^{km(B)} \right) \right] \right\}, \quad (28)$$

where the associated metrics $g^{ik(A)}$ and $g^{ik(B)}$ are

$$g^{im(A)} = U^i U^k + \frac{\varepsilon_\perp}{\varepsilon_\parallel} \Delta^{im} + \frac{(\varepsilon_\perp - \varepsilon_\parallel)}{\varepsilon_\parallel} X^i_{(r)} X^m_{(r)}, \quad (29)$$

$$g^{im(B)} = U^i U^k + \frac{\mu_\perp}{\mu_\parallel} \Delta^{im} + \frac{(\mu_\perp - \mu_\parallel)}{\mu_\parallel} X^i_{(r)} X^m_{(r)}. \quad (30)$$

Here $X^k_{(r)} = \delta^k_r \sqrt{N(r)}$ is the radial tetrad space-like four-vector and $\Delta^k_m \equiv \delta^k_m - U^k U_m$ is the projector. The representation (28) is a particular case of the decomposition (19) with

$$G_{(A)(A)} + G_{(B)(A)} = 1, \quad G_{(A)(B)} + G_{(B)(B)} = 0, \quad G_{(A)(B)} = G_{(B)(A)} = \gamma,$$

$$\frac{1}{\hat{\mu}} = \varepsilon_\parallel, \quad \frac{1}{\gamma} = 1 - \frac{\varepsilon_\parallel \mu_\perp}{\varepsilon_\perp \mu_\parallel}. \quad (31)$$

Mention that $g^{ik(B)}$ can be obtained from $g^{ik(A)}$ by a formal replacement of the symbol ε by the symbol μ .

3.2. Optical metrics

In the approximation of geometrical optics [21] the potential four-vector A_l and the field strength tensor can be represented as

$$A_l = a_l e^{i\Phi}, \quad F_{mn} = \nabla_m A_n - \nabla_n A_m = i(k_m A_n - k_n A_m), \quad (32)$$

respectively, where Φ is the phase, a_l is a slowly varying amplitude, and k_m is a wave four-vector:

$$k_m \equiv \nabla_m \Phi. \quad (33)$$

In the leading-order approximation the Maxwell equations reduce to

$$C^{ilmn} k_l k_m a_n = 0. \quad (34)$$

This system of linear equations with respect to a_n admits non-trivial solutions, when four components of k_l satisfy the Fresnel equation (see, e.g., [15] for details), which is usually called the dispersion equation. Using the decomposition (28) with (29) and (30) in (34) after some algebra we obtain the dispersion equations for two different principal cases.

(i). The first case relates to the propagation of electromagnetic wave with non-vanishing longitudinal polarization, i.e., with $a_r \neq 0$. The Fresnel equation admits the solution

$$\frac{\omega^2}{c^2} + k_r k^r + \frac{1}{\varepsilon_\parallel \mu_\perp} (k_\theta k^\theta + k_\varphi k^\varphi) = 0, \quad (35)$$

where $\omega \equiv U^i k_i$ is a frequency. This dispersion relation can be reconstructed as an eikonal-type equation

$$g^{im(A)} k_i k_m = 0, \quad (36)$$

where $g^{im(A)}$ is the first associated metric (29).

(ii). The second case is characterized by $a_r = 0$. The Fresnel equation admits the solution

$$\frac{\omega^2}{c^2} + k_r k^r + \frac{1}{\varepsilon_{\perp} \mu_{||}} (k_{\theta} k^{\theta} + k_{\varphi} k^{\varphi}) = 0, \quad (37)$$

which can be rewritten as

$$g^{im(B)} k_i k_m = 0, \quad (38)$$

where $g^{im(B)}$ is the second associated metric (30).

When the electromagnetic wave propagates along the radial direction, i.e., when $k_{\theta} = k_{\varphi} = 0$ and $k_r k^r = -k^2$, the dispersion relations (35) and (37) give the same result, $\omega^2 = k^2 c^2$, providing the phase velocity to be equal to speed of light in standard vacuum. The velocity of propagation in the transversal directions (i.e., when $k_r = 0$) depends on the polarization: when $a_r \neq 0$ the phase velocity of the wave is $V_{(1)} = \sqrt{\frac{\varepsilon_{\perp}}{\varepsilon_{||}}}$, when $a_r = 0$ it is $V_{(2)} = \sqrt{\frac{\mu_{\perp}}{\mu_{||}}}$.

4. Photon dynamics

The associated metrics obtained in Subsection 3.1 are the optical ones, i.e., they satisfy the eikonal equation. This means that the motion of a photon influenced by the non-minimal interaction can be described by null geodesic lines in the effective space-times with optical metrics $g^{im(A)}$ or $g^{im(B)}$, depending on polarization. Three metrics: the metric of space-time (3), the first and second optical metrics (29) and (30), can be written in general form

$$ds_{(E)}^2 = N(r) dt^2 - \frac{dr^2}{N(r)} - r^2 Y_{(E)}(r) (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (39)$$

where the index E takes the values 0, A , B and $Y_{(0)}(r)=1$. For the sake of simplicity we consider below the monopole with $M = 0$. It is characterized by three interesting features: first, $N(r) \geq 1$ for arbitrary r , thus, $N(r)$ does not reach zero; second, $N(r)$ reaches its maximum at $r = a$, the maximal value being $N_{(max)} = 1 + \sqrt{\frac{\kappa}{16q}}$; third, $N(0) = 1$ and curvature invariants are regular in the center $r = 0$. For this particular model we have

$$Y_{(A)}(r) = \frac{1 - 11\xi + 37\xi^2 + \xi^3}{1 + 9\xi - 7\xi^2 + \xi^3}, \quad (40)$$

$$Y_{(B)}(r) = \frac{1 + 9\xi - 7\xi^2 + \xi^3}{1 - 7\xi + 9\xi^2 + \xi^3}, \quad (41)$$

where ξ is dimensionless positive quantity $\xi = \kappa q / r^4$. The polynomial $1 - 11\xi + 37\xi^2 + \xi^3$ has no real positive zeros, i.e., $Y_{(A)}(r) \neq 0$. The equalities $Y_{(A)}(r) = \infty$ and simultaneously $Y_{(B)}(r) = 0$ take place, when $1 + 9\xi - 7\xi^2 + \xi^3 = 0$, i.e., at $\xi = \xi_1 \approx 1.85$ and $\xi = \xi_2 \approx 5.25$. Finally, $Y_{(B)}(r) = \infty$, when $\xi = \xi_3 \approx 0.19$ and $\xi = \xi_4 \approx 0.54$. Thus, despite the metric $ds_{(0)}^2$ is regular everywhere, the first and second optical metrics have singular points, when the angular functions $Y_{(A)}(r)$ and $Y_{(B)}(r)$ vanish. Physical interpretation of such (dynamic) singularities will be done in the next Subsection in the course of description of radial, tangent and inclined particle motion.

4.1. Photon trajectories

Let us consider null geodesic lines in the effective space-times with optical metrics (29) and (30). The geodesic equations have the standard form

$$\frac{d^2 x^k}{d\tau^2} + \Gamma_{jl(E)}^k \frac{dx^j}{d\tau} \frac{dx^l}{d\tau} = 0, \quad (42)$$

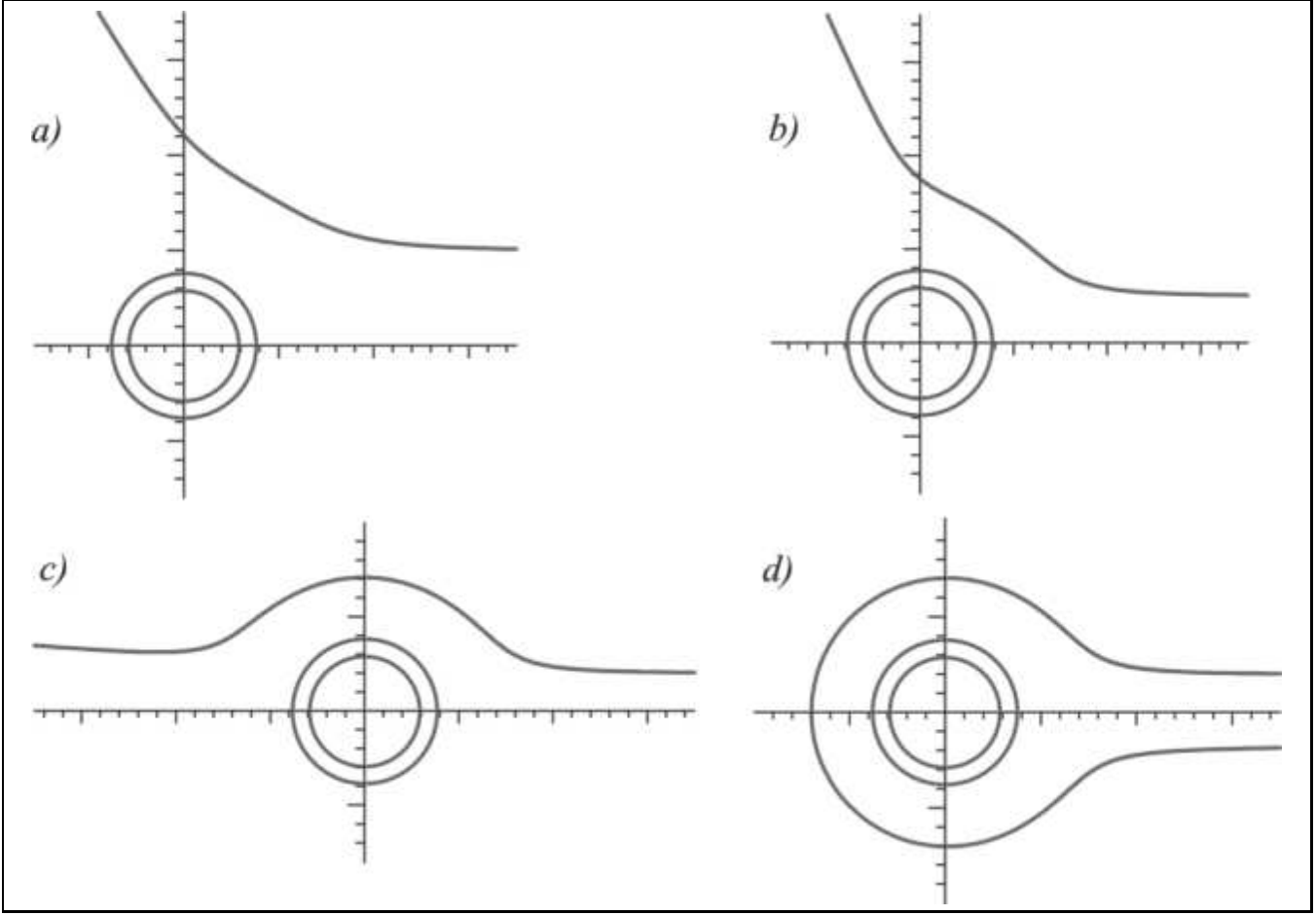


Figure 1: Rays for optical A-metric at the impact parameter $J > J_{\text{crit}}^{(A)} \approx 0.794659$: (a) $J = 2$, (b) $J = 1$, (c) $J = 0.8$, (d) $J = 0.79466$.

where the Christoffel symbols $\Gamma_{jl(E)}^k$, $E = 0, A, B$ are constructed on the base of the corresponding metric. As in classical case (see, e.g., [22]) the photon moves in the plane, and one can choose this plane as the equatorial one, $\theta = \pi/2$. Other integrals have the form

$$\frac{d\varphi}{d\tau} = \frac{J}{r^2 Y_{(E)}(r)}, \quad \frac{dt}{d\tau} = \frac{1}{N(r)}, \quad (43)$$

$$\left(\frac{dr}{d\tau}\right)^2 = 1 - \frac{J^2 N(r)}{r^2 Y_{(E)}(r)}, \quad (44)$$

providing the relation

$$ds_{(E)}^2 = \left[g_{(E)mn} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} \right] d\tau^2 = 0, \quad (45)$$

where the quantity J is the impact parameter. As a consequence of (43) and (44) the following relation takes place:

$$\left(\frac{dr}{d\varphi}\right)^2 = r^2 Y_{(E)}(r) \left[\frac{1}{J^2} r^2 Y_{(E)}(r) - N(r) \right]. \quad (46)$$

The formulas (43)-(46) are the non-minimally modified formulas for the well-known integrals of motion [22] (they coincide with them, when $Y_{(E)}(r) = 1$ only). Let us emphasize seven interesting details of these modified integrals of motion.

(I) Radial motion is non-sensitive to the non-minimal coupling. Indeed, when $\theta = \pi/2$ and $\varphi = \text{const}$, the parameter J vanishes and the relation $\frac{dr}{d\tau} = \pm 1$ is valid for each $E = 0, A, B$. In this case the velocity of photon coincides with speed of light in standard vacuum.

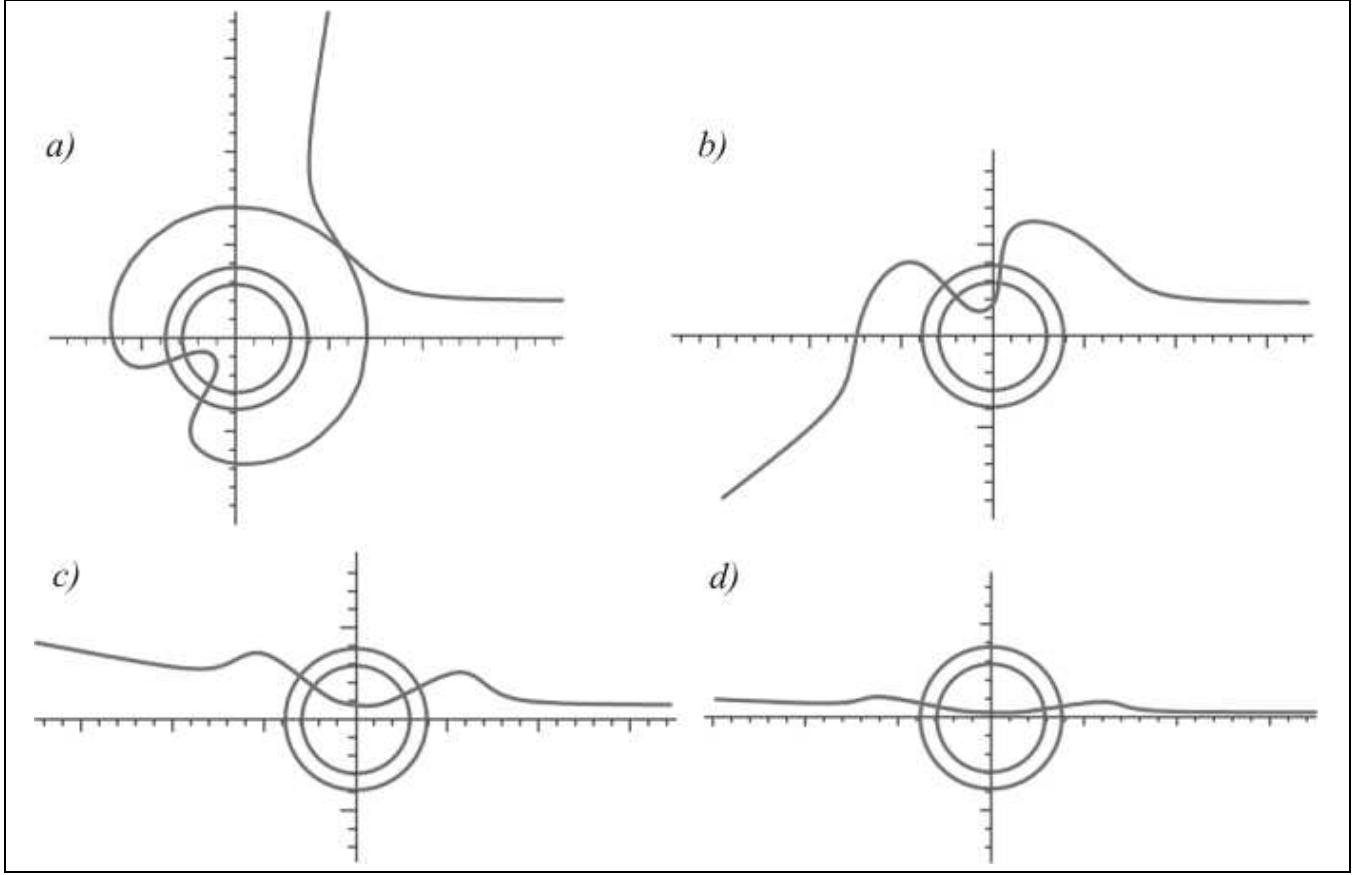


Figure 2: Rays for optical A-metric at the impact parameter $J < J_{\text{crit}}^{(A)} \approx 0.794659$: (a) $J = 0.7941$, (b) $J = 0.7$, (c) $J = 0.3$, (d) $J = 0.1$.

(II) When the motion is not pure radial (inclined), the condition $\frac{dr}{d\varphi} = 0$ describes the extrema of the curve $r(\varphi)$, giving the distance of the closest approach. When the curvature interaction is absent, i.e., $q = 0$, we obtain from (46) the standard condition for the minimal distance r_0 : $r_0^2 - J^2 N(r_0) = 0$. In the non-minimal model $Y_{(A)}$ and $Y_{(B)}$ are the functions of radius (see (40) and (41)), and the set of solutions to the equation $Y_{(E)}(r^2 Y_{(E)} - J^2 N) = 0$ is much more sophisticated than in the minimal model.

(III) Formally speaking, the optical metrics become non-Lorentzian, i.e., are of the signature $+-++$, when $Y_{(E)}(r)$ is negative, and are singular at the points, where $Y_{(E)}(r) = 0$. From the dynamic point of view the negative values of $Y_{(E)}(r)$ are, nevertheless, admissible. Indeed, the angular integral of motion in (43) remains consistent, but the direction of rotation becomes opposite; the right-hand-sides of the equations (44) and (46) remain positive, as it should be.

(IV) The functions $Y_{(E)}(r)$ change the sign at the points, where $Y_{(E)}(r) = 0$ or $Y_{(E)}(r) = \infty$. The first possibility can be realized for the B-ray only, when $\xi = \kappa q/r^4$ takes the values 1.85 or 5.25. In this case the equation (46) gives the condition $\frac{dr}{d\varphi} = 0$, indicating the point of extremum for the curve $r(\varphi)$ (see item III). The conditions $Y_{(E)}(r) = \infty$ can be realized at $\xi = 1.85$ and $\xi = 5.25$ for the A-ray, and at $\xi = 0.19$ and $\xi = 0.54$ for the B-ray. At these points the angular frequency $\frac{d\varphi}{d\tau}$ vanishes (see (43)) and the radial velocity $\frac{dr}{d\tau}$ coincides with speed of light in the standard vacuum (see (44)). Since the relation $\frac{dr}{d\varphi} = \infty$ holds at these points, the rays are directed radially.

(V) The relation $\frac{dr}{d\varphi} = 0$ is self-contradictory, when $Y_{(E)}$ is negative (see (44)). This means that tangent waves can not propagate, and we can indicate this zone as inaccessible one for them. For the tangent A-ray, there is only one inaccessible zone at $1.85 < \xi < 5.25$, for the tangent B-ray the second one appears at $0.19 < \xi < 0.54$. The corresponding boundary spheres can be indicated as trapped surfaces (see, e.g., [18, 19] for details).

(VI) For A- and B-rays there are specific “critical” values of the impact parameters

$$J_{\text{crit}}^{(A),(B)} = \max \left(\frac{N}{r^2 Y_{(A),(B)}} \right).$$

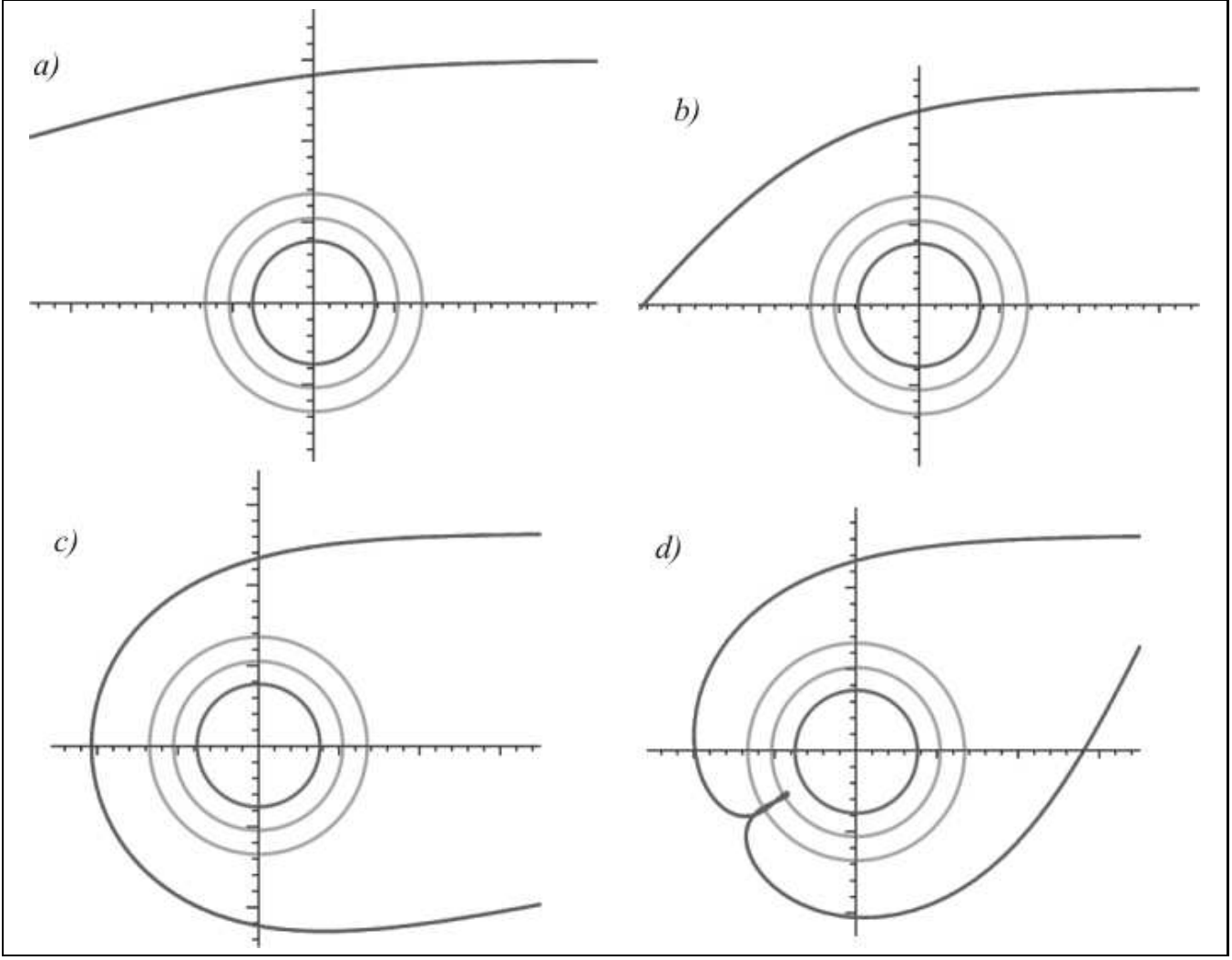


Figure 3: Rays for optical B-metric at the impact parameter $J > J_{\text{crit}}^{(B)} \approx 5.29385$: (a) $J = 6$, (b) $J = 5.4$, (c) $J = 5.294$, and (d) $J < J_{\text{crit}}^{(B)}$, $J = 5.29$.

If $J > J_{\text{crit}}$, the corresponding rays do not reach the surface with $Y_{(E)}(r) = \infty$. Mention that in minimal case such “critical” values of the impact parameters do not exist.

(VII) Since the space-time metric of the non-minimal monopole is regular, i.e., $N(r) \neq 0$ and $N(r) \neq \infty$, in the $t - r$ cross-section there is no singularities.

4.2. Numerical modeling of the photon trajectories

Pictures presented in the Fig. 1-4 illustrate the dependence $r(\varphi)$ for rays, which are null geodesics of the optical metrics $g_{(A)}^{ik}$ and $g_{(B)}^{ik}$. The rays start from infinity, the impact parameter being equal to the height of the starting point of curve. We put $\kappa = 1$, $q = 10$. In the Fig. 1, 2 two circles relate to $r = R_1 \equiv \sqrt[4]{\kappa q/1.85}$ and $r = R_2 \equiv \sqrt[4]{\kappa q/5.25}$; in the Fig. 3, 4 three circles relate to $r = R_3 \equiv \sqrt[4]{\kappa q/0.19}$, $r = R_4 \equiv \sqrt[4]{\kappa q/0.54}$, and $r = R_1$, respectively. The picture (a) in the Fig. 2 contains one self-intersection point, the picture (d) in the Fig. 3 and the picture (a) in the Fig. 4 contain two self-intersection points. The region inside the circle $r = R_1$ is inaccessible zone for the inclined B-rays. These figures show that the non-minimal Dirac monopole can be considered as a scattering center for inclined rays. Scattering laws for A- and B-rays are different confirming the birefringent character of non-minimal interaction.

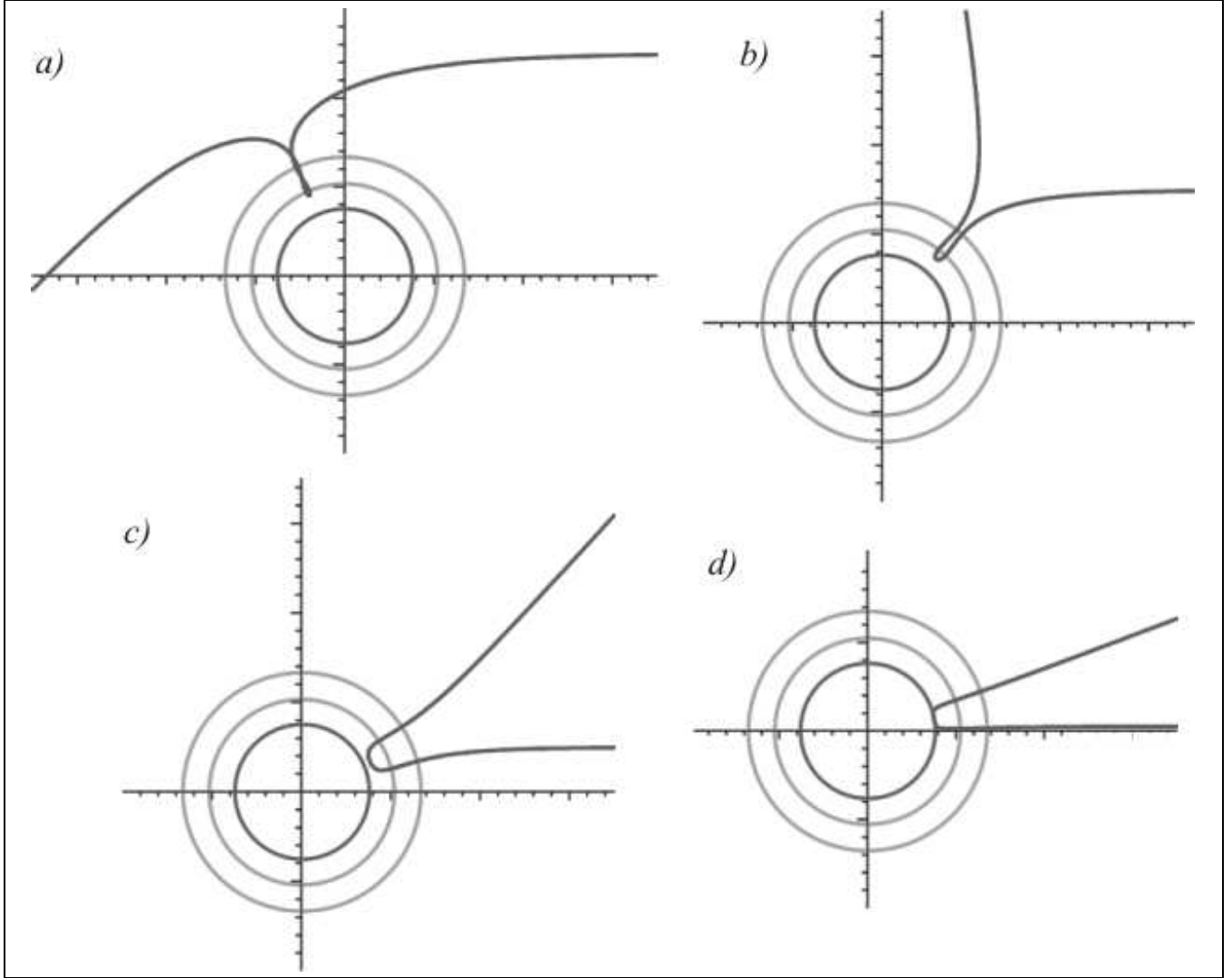


Figure 4: Rays for optical B-metric at the impact parameter $J < J_{\text{crit}}^{(B)} \approx 5.29385$: (a) $J = 5$, (b) $J = 3$, (c) $J = 1$, (d) $J = 0.1$.

5. Conclusions

Qualitative and numerical analysis of the photon orbits in the vicinity of non-minimal Dirac monopole with regular metric has demonstrated the following interesting features.

1. Propagation of the electromagnetic waves in the vicinity of non-minimal Dirac monopole is characterized by birefringence, induced by curvature, i.e., the phase velocities of waves depend on their polarization. Two different optical metrics should be introduced to describe two principal states of polarization.

2. The metric of the non-minimal Dirac monopole, obtained and discussed in this paper, is regular, thus, all the singularities of the optical metrics have a dynamic origin and are supported by the non-minimal (curvature induced) interaction of the gravitational and electromagnetic fields.

3. The points of self-intersection, the points of the closest approach, the reverse points, etc. in the photon trajectories can be recognized and catalogued for different combinations of the values of the impact parameter and the parameter of non-minimal coupling. We discussed here only the principal pictures.

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